



Enhancements for technique of stable low cycle fatigue crack propagation simulation for gas turbine engine

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Abstract

Propagation of low cycle fatigue cracks in main parts of an aviation gas turbine engine can cause sudden engine destruction during operation with catastrophic consequences. Accurate prediction of an engine low cycle fatigue durability can be accessed using finite element simulations. Main aspects of the applied approach for low cycle fatigue stable crack propagation finite element simulations in 3D specification for main parts of an aviation gas turbine engine are described in the article. These include applied formulation and procedure, descriptions of applied models and assumptions, parameters of generated grids, some results of grid convergence studies and simulation results for turbine discs. The significance of some aspects for solution accuracy such as along crack front grid resolution is demonstrated. Advancements to the applied approach are proposed which include solution under-relaxation preventing instability when large crack propagation steps are used and circular correction improving crack path prediction accuracy for bodies of revolution. The improvements result in the overall simulation accuracy increase and effort decrease up to 50%.

Keywords AGTE · Low cycle fatigue · Stable crack growth · Modeling aspects

1 Introduction

Main parts of aviation gas turbine engines (AGTE) such as rotor discs are subjected to severe cyclic thermal–mechanical loading in operation. Under such conditions, low cycle fatigue cracks appear and propagate in the parts causing a significant threat of sudden destruction of an engine. This makes accurate prediction of low cycle fatigue durability of main AGTE parts vital.

Major steps in description of a crack mechanical behavior were done with development of linear elastic fracture mechanics (LEFM) theory [1–5]. The prediction of low cycle fatigue durability became available with formulation of different empirical dependencies from which the most widespread became Paris–Erdogan equation [6]. Crack

propagation simulations described in this article were implemented according to the industry used approach which main statements are described in papers [7–10]. The applied modeling approach for stable crack propagation from initial flaw in parts of an AGTE has semi-empirical nature which is based on LEFM theoretical statements [1–5], empirical Paris–Erdogan equation [6], fractographic analysis data for crack surfaces in AGTE parts [11] and metal science vision of fragmented material structure in vicinity of low cycle fatigue crack front [12]. Deformed state of solid bodies is evaluated using finite element method (FEM) [13, 14].

Some results of stable crack propagation simulations for main parts of an AGTE are contained in the article. Existing difficulties related to the modeling approach are highlighted. Also descriptions of new modeling aspects used in the current work are provided. These are essential for accuracy improvement and effort decrease of the simulations.

Several alternative methods for simulation of crack propagation are listed in [15]. Focus of comparison in the pointed article is given to the two main methods available in commercial codes such as ABAQUS [16]: FEM and extended FEM (XFEM) [17, 18]. Overview of the XFEM approach and its applications is provided in [19]. It has the main significant advantage – no need for remeshing during simulation

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which has a huge potential for effort savings. But it also has disadvantages such as large FE model size since fine grid is needed along the whole crack propagation expected path and etc., which makes the potential effort saving questionable. Comparison of the applied industry standard approach to the XFEM approach is outside the scope of the current study.

2 Problem statement and methods

Main parts of an AGTE of which the majority are rotor discs are considered in 3D in isolated manner or in scope of a subassembly depending on connection technology used for the neighboring parts and position of focus of interest on a particular part. To decrease the FE model grid size, a representative sector of the subassembly is considered with periodic boundary conditions on sides of the sector:

$$u(x) = u(Qx) \text{ on } \Omega. \quad (1)$$

Here u displacement vector of a body point; x initial position vector of a body point; Q transformation tensor which rotates position vectors around rotor axis by the sector angle; Ω the sector periodic boundaries' surface.

Following loads are considered which are listed in descending order of its impact on to the parts' stressed state: centrifugal load; temperature field; pretension stresses from bolts and press fits; gas pressure on the subassembly wetted surface; radial, axial and tangential forces from the neighboring subassemblies. The temperature and pressure fields are obtained in a separate transient conjugate heat transfer (CHT) analysis of the AGTE operating on a typical flight cycle. Descriptions for the CHT analysis are outside the scope of the current article.

Blades are considered as deformable bodies. Contact interaction of rotor discs with blades and other parts included in the considered subassemblies is set-up as mechanical sliding contact with friction.

Low cycle durability estimate for a part is done based on the following statements with the listed chain of steps:

1. Transient static temperature dependant mechanical solution is done for a considered subassembly. Critical stressed spots are located on the parts' surfaces. Maximum range of stress tensor first eigenvalue (σ_1) during the typical flight cycle serves as a criterion for selection of a critical spot. Maximum σ_1 value in the spot corresponds to the estimated time point during the typical flight cycle (condition "M").
2. Load cycle "0-M-0" is considered further in the simulation. Here condition "0" corresponds to cold inactive AGTE.

3. Initial flaw is introduced into the part model as a zero-width cut at the critical spot location. Initial flaw size is taken equal to the maximum possible defect which is not detected during control after manufacturing. Initial flaw shape depends on the part manufacturing and the surface machining technology. It is defined based on qualified engineering insight.
4. Initial flaw plane is positioned normal to σ_1 vector.
5. Stable crack propagation process is simulated as a series of solutions. During the series a crack length is increasing according to statements provided further in Sect. 2.2.
6. The crack is said to reach the top boundary of its stable propagation process when it starts growing $2 \mu\text{m}$ per load cycle or more [7–11]. After reaching the boundary or after confirming the prescribed durability the simulation is stopped.

2.1 Solid behavior

Solid parts mechanical behavior is modeled using Lagrangian FEM implementation in Abaqus [16] commercial code. Implicit integration scheme is used. Material behavior is considered isotropic linear elastic. Large displacement static formulation for temperature dependant mechanical problems is applied [20]:

$$\nabla \sigma : I + f = 0, \quad (2)$$

$$\sigma = C : (\varepsilon - \varepsilon_T), \quad (3)$$

$$\varepsilon = 0.5(I - b^{-1}), b = FF^T, \quad (4)$$

$$F = I + \nabla u, \quad (5)$$

$$\varepsilon_T = \alpha(T - T_{\text{ref}}). \quad (6)$$

Here ∇ gradient operator; σ Cauchy stress tensor; F body force vector; I unity tensor; C elasticity tensor; ε Euler-Almansi strain tensor; F displacement gradient tensor; ε_T thermal strain tensor; α thermal expansion coefficients tensor; T body point temperature; T_{ref} reference temperature at which thermal strain is considered zero.

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), i, j, k, l = 1, 2, 3, \quad (7)$$

$$\lambda = E\nu / [(1 + \nu)(1 - 2\nu)], \mu = E / [2(1 + \nu)], \quad (8)$$

$$\alpha_{ij} = \alpha \delta_{ij}. \quad (9)$$

Here C_{ijkl} tensor C components; $E = E(T)$, $\nu = \nu(T)$ temperature dependent material elastic modulus and Poisson's ratio consequently; λ , μ material Lamé parameters;

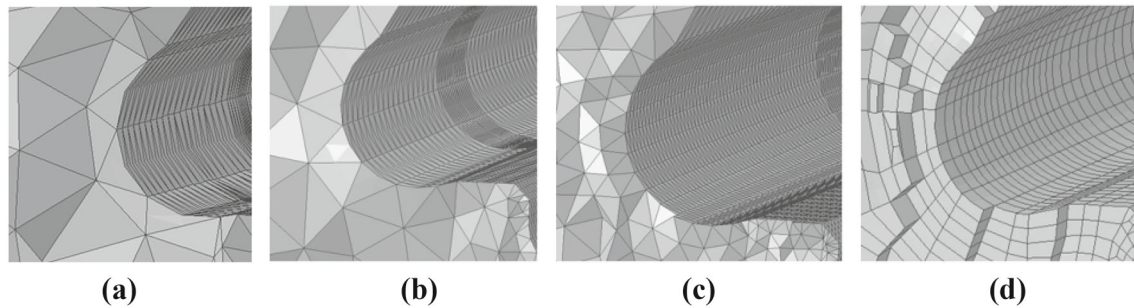


Fig. 1 Grid convergence study

α_{ij} tensor α components; $\alpha = \alpha(T)$ temperature dependent material isotropic thermal expansion coefficient; δ_{ij} Kronecker delta.

2.2 Crack growth model

Main equations allowing simulation of stable crack propagation from an initial flaw are the following [7, 8, 10]:

$$\Delta l = B \cdot (\Delta K_I / E)^2 \cdot N, \quad (10)$$

$$\theta = -\arccos[(3K_{II}^2 + K_I \sqrt{K_I^2 + 8K_{II}^2}) / (K_I^2 + 9K_{II}^2)]. \quad (11)$$

Here Δl crack propagation in a point of crack front in direction normal to the front during N loads cycles; $B = 10$; K_I , K_{II} stress intensity factors for the 1st and 2nd crack propagation modes consequently; ΔK_I range of K_I in a load cycle; θ crack propagation vector slope angle.

New crack front is formed by hodograph of vectors defined by $(\Delta l, \theta, 0)$ coordinates in local cylindrical coordinate systems (R, Θ, Z) with origin in the considered current crack front point with local Z axis tangent to the current front curve, and $\Theta = 0$ line going through the corresponding points of the current and the previous crack fronts.

Crack propagation simulation is done starting from an initial flaw in “step by step” manner by building a new crack front relative to the previous crack front. N value is taken with respect to an assumption that estimated K_I , K_{II} and ΔK_I values retain nearly constant during N load cycles. Allowed discrepancy from constant of the listed variables depends on balance between desired solution accuracy and spent effort. Usually the discrepancy should be not more than 5%.

2.3 FE Grid

2.3.1 FE models for location of stress spots

To get advantage of automatic grid generation continuum 2nd order tetra elements (C3D10 in Abaqus terminology [21]) were used. Converged stress-strain solution on stress con-

centrators of the considered solid bodies was guaranteed by application of special rules to the generated grids. Some of the applied rules:

1. Aspect ratio of elements adjacent to body surfaces should be close to unity.
2. Number of elements across fillet surface should be five or more.
3. Number of elements across part wall should be four or more.

The rules are supported by preliminary grid convergence studies done for typical parts. Meshing based on rules validated by grid convergence studies done for typical parts and stress concentrators prevents the need to repeat such study for each new part. Illustration for such study is provided in Fig. 1. In Fig. 1b the shown grid is generated according to the rules. In Fig. 1a twice coarser grid is shown, in Fig. 1c—twice finer. In Fig. 1d grid of 2nd order hexa elements of reduced integration (C3D20R in Abaqus terminology [21]) is shown. Density of the grid is comparable to the grid from Fig. 1c.

First eigenvalue of stress tensor in the stressed spot on the concentrator obtained on the grid from Fig. 1(b) differs from the solution on the grid from Fig. 1a 0.63%, from solution on the grid from Fig. 1c—0.08%, from solution on the grid from Fig. 1d—2.0%. The provided comparison justifies an assumption that application of the listed meshing rules to the studied parts guarantees convergence of the maximum stress value on a fillet with discrepancy less than 2%. Analogical comparisons were done for other rules and corresponding concentrator types.

In Fig. 2a, b typical FE model of a disc is shown. Representative sector angle is 10° in this case. The grid contains 140 k tetra elements and 218 k nodes.

2.3.2 Focused grid around crack front

In Fig. 2c fragment of the disc grid used for crack propagation solution is shown. The crack is highlighted by bold line, refined grid around the crack front—by dark shading of

Fig. 2 Grid of a disc (a), fragment of the disc grid near blade seat (b), slice of the disc grid across crack (c)

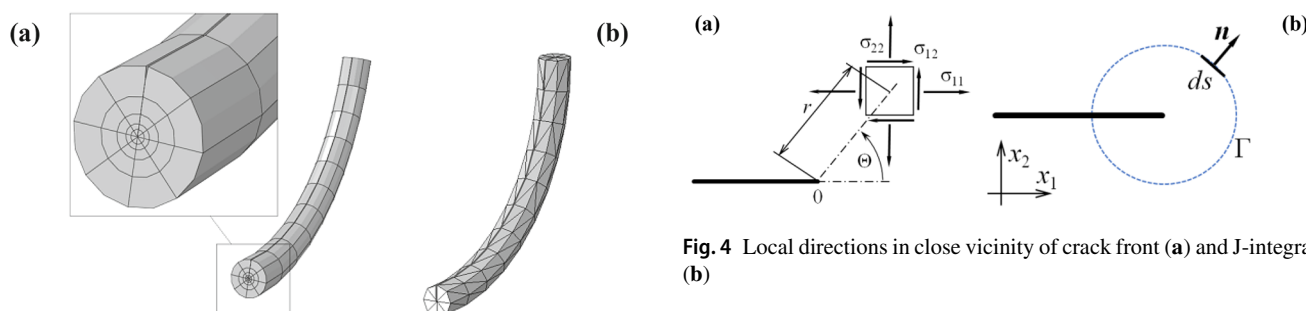
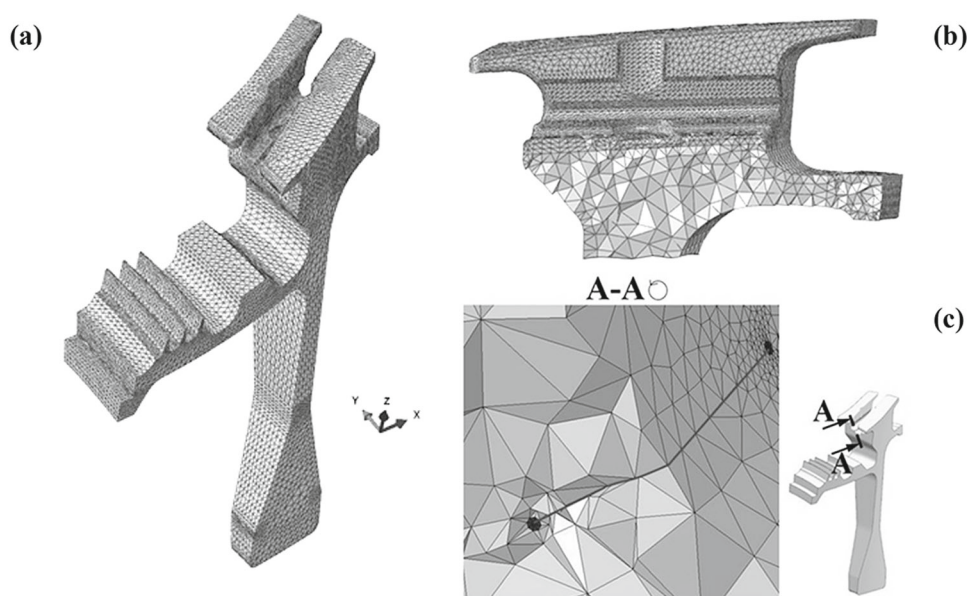


Fig. 3 Example of submodel grid (a) and corresponding part model fragment (b)

element faces. A disc sector grid with introduced crack and corresponding local refinement usually contains up to 2 M tetra elements and 2.5 M nodes depending on crack length.

Stress–strain condition in vicinity of crack front is evaluated on a sub-model of 2nd order continuum hexa elements (C3D20 in Abaqus terminology [21]). The submodel uses special focused grid, example of such grid for an initial crack front is shown in Fig. 3a. Corresponding fragment of the part model around crack front is shown in Fig. 3b. Nodes on outside surface of the sub-model and the fragment of the part model are coincident in space for accurate application of submodel Dirichlet boundary conditions as prescribed node displacements from structural solution for the part.

Stress fields for plain strain condition in close vicinity of a crack front are described by the following equation [3–5]:

$$\sigma_{ij} = \sum_k [K_k / \sqrt{2\pi r}] f_{ijk}(\Theta). \quad (12)$$

Here σ_{ij} Cauchy stress tensor components; $i, j = 1, 2$ —local coordinate system axis numbers; K_k —stress intensity

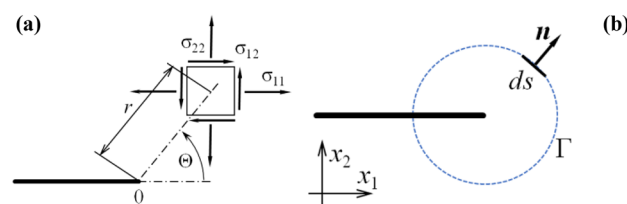


Fig. 4 Local directions in close vicinity of crack front (a) and J-integral (b)

factor for the k th mode; $k = I, II$ since only K_I and K_{II} are considered in the current study; r distance from the considered point on crack front; $f_{ijk}(\Theta)$ function depending on angle Θ . The descriptions are illustrated in Fig. 4.

In Fig. 4 section of a part volume in close vicinity of crack front is shown. The section plane normal is tangent to the crack front in the considered point. The crack is shown by bold line.

Stress intensity factors are evaluated from J-integral [3–5, 22, 23] values which are calculated along path around crack tip in the considered section. Such path is represented by one ring of submodel elements adjacent to the considered node.

$$K_k = G_k E / (1 - \nu^2), \quad (13)$$

$$G_k = J_k = \int_{\Gamma} (W dx_j - (\sigma \cdot \mathbf{n}) \cdot (\partial \mathbf{u} / \partial x_1) ds), \quad W = \int_0^{\epsilon} \sigma : d\epsilon. \quad (14)$$

Here G_k strain energy release rate for the k th mode; J_k J -integral for the k th mode; $k = i, i \neq j; i, j = 1, 2$ —local coordinate system axis numbers; W strain energy density; $\mathbf{n}$ normal vector to the path Γ ; s coordinate along the path. The descriptions are illustrated in Fig. 4.

Internal ring of sub-model elements (Fig. 3a) has $\sim 1 \mu\text{m}$ radius and consists of degraded 2nd order hexa elements. Edge length of the elements on the crack front is zero. Mid-side nodes on edges adjacent to the crack front are moved to a position on 1/4 of the corresponding edge from the zero-length edge. This produces singular stress fields in the elements which are proportional to $1/\sqrt{r}$. By this way the theoretical solution described by Eq. 12 is reproduced in vicinity of the crack front.

Submodel grid and corresponding part model fragment grid parameters sufficient for evaluation of stress intensity factors' values with discrepancy less than 5% were obtained during grid convergence study. Following rules were finally applied to submodel grid:

1. Number of elements in one ring along circumference, $n_\Theta = 8$.
2. Number of element rings in radial direction, $n_R = 7$.
3. Radial to circumferential aspect ratio of all submodel elements should be the same, defined by n_R and n_Θ values.
4. Radial to axial aspect ratio of all submodel elements should be the same and not more than 20.
5. Submodel resolution of 10–20 elements along the initial crack front was considered sufficient. This parameter together with the other listed rules defines the submodel outside radius. For the following crack fronts the radius value and radial to axial aspect ratio of elements are kept constant.
6. For the corresponding part model fragment one ring of tetra elements is considered sufficient.

2.4 Long crack modeling aspects

2.4.1 Crack propagation under-relaxation

Crack propagation scheme described in Sect. 2.2 is explicit in nature which means that it is conditionally stable depending on Δl size. Simulation of crack propagation to a large length ($l_{\max}/l_0 > 10 \dots 20$, l_0 initial flaw length; $l_{\max}$ crack length corresponding to the final simulation step) is always desired to be done with large Δl size to decrease effort for the simulation. In this case solution smoothing can help avoiding crack front fluctuations which are appearing with increase of Δl . Smoothing approach applied in this study is under-relaxation:

$$K^{[u]} = K^{[i]} \cdot \text{URF} + K^{[i-1]} \cdot (1 - \text{URF}). \quad (15)$$

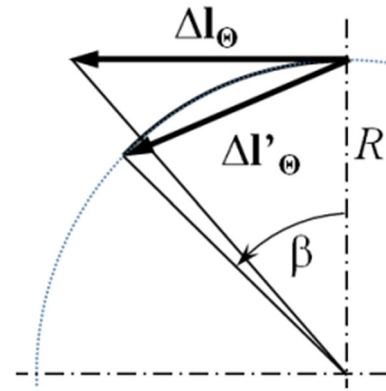


Fig. 5 Circular correction

Here: $K = K_I$ or K_{II} ; $\text{URF} \in [0, 1]$ [0, 1]—under-relaxation factor; indexes mean the following: $[u]$ under-relaxed value; $[i]$, $[i - 1]$ value from current and previous solution steps correspondingly.

2.4.2 Circular correction

As it is stated in Sect. 2.2 crack propagation in a point on current crack front is described by Δl and θ parameters. Intention to increase Δl size can lead to significant inaccuracy in resulting crack path and consequent evaluated durability. This is especially visible in case if crack propagates in circumferential direction in a part which is a body of revolution, since Δl line segment is positioned tangent but not as a circle chord. The inaccuracy is avoided by correction of tangential component of Δl vector:

$$\Delta l = \Delta l_R \mathbf{r} + \Delta l_\Theta \boldsymbol{\tau} + \Delta l_Z \mathbf{z}, \Delta l' = \Delta l'_R \mathbf{r} + \Delta l'_\Theta \boldsymbol{\tau} + \Delta l_Z \mathbf{z}, \quad (16)$$

$$\Delta l'_R = \Delta l_R - R[1 - \cos(\beta)], \Delta l'_\Theta = R \sin(\beta), \quad (17)$$

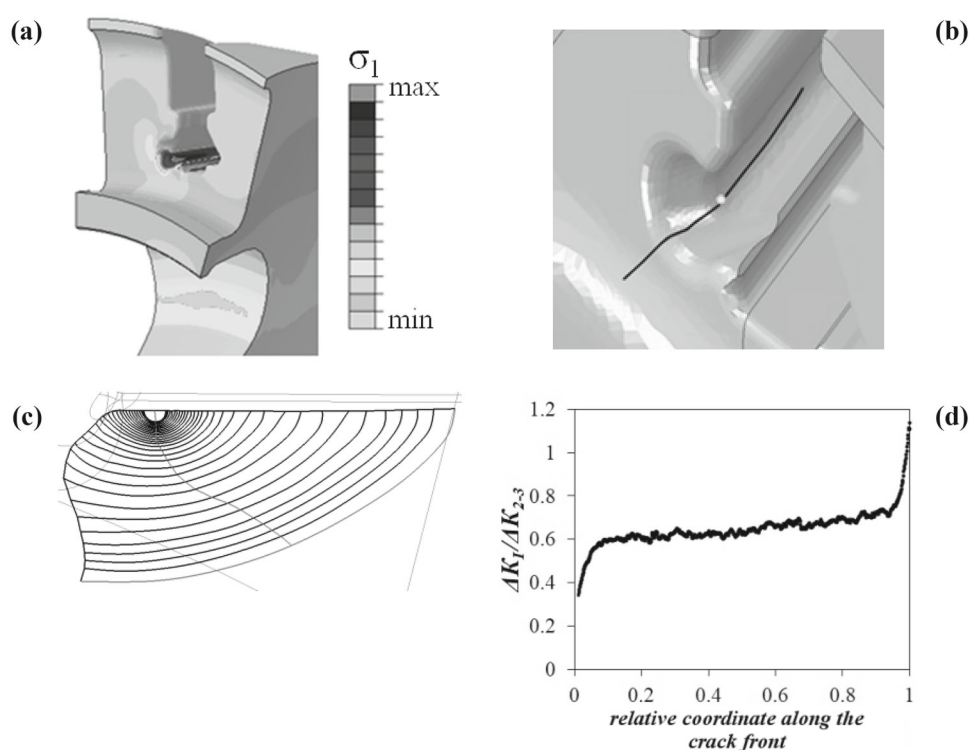
$$\beta = \Delta l_\Theta / R. \quad (18)$$

Here Δl , $\Delta l'$ initial and corrected crack propagation vectors of a point on front consequently; $\mathbf{r}$, $\boldsymbol{\tau}$, $\mathbf{z}$ local basis unit vectors in radial, tangential and axial directions consequently; Δl_R , $\Delta l'_R$, Δl_Θ , $\Delta l'_\Theta$, Δl_Z —coordinates of Δl and $\Delta l'$ vectors in $(\mathbf{r}, \boldsymbol{\tau}, \mathbf{z})$ basis; R distance from the considered crack front point to the rotor axis.

Illustration to the correction is given in Fig. 5. It assumes that real crack propagation in tangential direction has circular arc but not line segment shape. Length of the resulting arc should be equal to the initial line segment. In Fig. 5:

$$\Delta l_\Theta = \Delta l_\Theta \boldsymbol{\tau}, \Delta l'_\Theta = \Delta l'_\Theta \boldsymbol{\tau} - R[1 - \cos(\beta)]\mathbf{r}. \quad (19)$$

Fig. 6 First main stress in vicinity of critical spot (a), developed crack on the disc surface (b), evaluated crack fronts on simulation steps (c), first stress intensity factor range along the last crack front (d)



3 Results and discussion

The approach described in Sect. 2 potentially allows accurate simulation of stable crack propagation in AGTE main parts such as rotor discs. Essential aspect of the simulation process is related to the number of nodes used to build new crack fronts on simulation steps. In the current study each submodel node on the current crack front was used to build a new front. Number of the used nodes depending on the current crack length varied from 10 at the initial flaw front up to ~4 k at latest fronts of comparably long cracks. Special application was developed in C++ [24] programming language to automate generation of submodel grids, stress intensity factors related data processing and build-up of new crack fronts on simulation steps making implementation of the approach practical. An example demonstrating importance of sufficient crack front longitudinal resolution is shown in Fig. 6.

In Fig. 6b initial flaw position is pointed by white spot. In Fig. 6c crack surface is represented by the series of crack fronts evaluated on simulation steps. The provided view corresponds to a direction producing the maximum crack surface projection area. In Fig. 6d a plot of first stress intensity factor range along the last crack front (shown in gray in Fig. 6c) is provided. Zero relative coordinate value corresponds to the bottom-left crack end in Fig. 6c. Here [7–11]:

$$\Delta K_{2-3} = E\sqrt{\Delta l_{2-3}/B}, \Delta l_{2-3} = 2\mu\text{m} \quad (20)$$

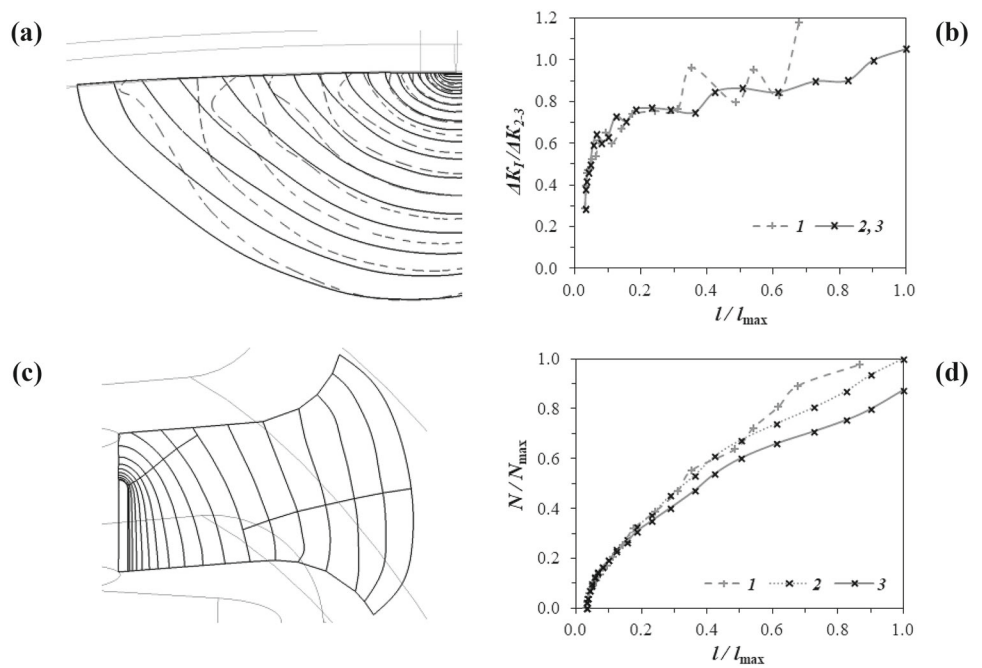
The curve shape presented in Fig. 6d can be explained by the stress field in vicinity of the critical spot. The top-right crack end in Fig. 6c during crack propagation is moving along the fillet of the blade seat and is always placed in a highly stressed area. The bottom-left crack end moves down along the disc side surface to a less stressed area. The condition justifies the stress intensity factor fast growth on the right side of the plot and decrease on the left side. In this case if new crack fronts on simulation steps were build using few nodes in central portion of current crack fronts as it is usually done when the process is implemented manually, there is a high risk of the predicted durability to be overestimated.

In Fig. 7a two series of crack fronts on solution steps are shown. Dashed lines perform a solution using moderately large Δl step without application of under-relaxation described in Sect. 2.4. Solid lines perform a solution with the same Δl step but with under-relaxation applied. In Fig. 7b $\Delta K_I(l)$ curves are shown corresponding to the compared solutions: 1—without under-relaxation; 2, 3—with under-relaxation. The numeration is retained in Fig. 7d.

In Fig. 7b, d: l —current crack length; $l_{\max}$ —maximum crack length for curves 1–3; $N_{\max}$ —maximum N value for curves 1–3.

Significant crack front fluctuations are observed for the solution without under-relaxation (free) while the under-relaxed solution remains stable. From the provided studies the following observation can be stated: stress intensity factors under-relaxation allows increasing Δl size from $(0.05 - 0.15)l$ to $(0.1 - 0.3)l$ without appearance of the growing

Fig. 7 Comparison of under-relaxed and free crack fronts on solution steps (a), $\Delta K_I(l)$ plot (b), durability recalculation path (c), $N(l)$ plot (d)



crack front fluctuations. The top range values correspond to smaller l (initial crack propagation stage), the bottom values—to larger l (long cracks).

$$d(\Delta K_k)/dN = (d(\Delta K_k)/dl) (dl/dN), k = I, II, \quad (21)$$

$$dl/dN \neq 0 \rightarrow d(\Delta K_I)/dN = 0 \text{ only if } d(\Delta K_I)/dl = 0, \quad (22)$$

$$K_{II} \approx 0 = \text{const} \rightarrow dK_{II}/dl = 0 \quad (23)$$

From Eqs. 21 and 22 it can be said that under-relaxation of ΔK_I doesn't influence the predicted durability only if $d(\Delta K_I)/dl = 0$. This means that after solution with under-relaxation recalculation of the predicted durability is needed for the evaluated crack front series. This is done using Eq. 10 along a path connecting corresponding points of the sequential crack fronts in the series. The path needs to be orthogonal to the crack fronts at intersection points. An example of such path is shown in Fig. 7c by bold black line intersecting the shown crack fronts. Since each crack front on a solution step is isochoric the path can be composed of several curves. In Fig. 7d curve 2 corresponds to the under-relaxed solution before durability recalculation, curve 3—after recalculation.

If the studied crack is propagating in the right direction on solution steps, then $K_{II} \approx 0 = \text{const}$. This means that under-relaxation of K_{II} has minimum influence on the solution.

4 Conclusions

Some results of stable crack propagation simulations for main parts of an AGTE were presented in the article. Industry standard approach was used for the simulations. Essential

aspects of the modeling process were highlighted including rule based sufficient quality grid generation, necessity of high longitudinal crack front resolution accompanied by the simulation automation to make it practical. Improvements to the applied approach were proposed and implemented which include solution under-relaxation and circular correction. The improvements result in the simulation accuracy increase and effort decrease up to 50%. Further development of the approach can include investigation of alternative criteria for location of critical spots on parts' surface and estimated time points during flight cycle.

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Data availability The data used to support the findings of this study is available from the corresponding author upon request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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